

Characterization of Co-Blockers for Simple Perfect Matchings in a Convex Geometric Graph

Chaya Keller and Micha A. Perles
Einstein Institute of Mathematics, Hebrew University
Jerusalem 91904, Israel

November 30, 2010

Abstract

Consider the complete convex geometric graph on $2m$ vertices, $CGG(2m)$, i.e., the set of all boundary edges and diagonals of a planar convex $2m$ -gon P . In [3], the smallest sets of edges that meet all the simple perfect matchings (SPMs) in $CGG(2m)$ (called “blockers”) are characterized, and it is shown that all these sets are caterpillar graphs with a special structure, and that their total number is $m \cdot 2^{m-1}$. In this paper we characterize the co-blockers for SPMs in $CGG(2m)$, that is, the smallest sets of edges that meet all the blockers. We show that the co-blockers are exactly those perfect matchings M in $CGG(2m)$ where all edges are of odd order, and two edges of M that emanate from two adjacent vertices of P never cross. In particular, while the number of SPMs and the number of blockers grow exponentially with m , the number of co-blockers grows super-exponentially.

1 Introduction

In this paper we consider convex geometric graphs (i.e., graphs whose vertices are points in convex position in the plane, and whose edges are segments connecting pairs of vertices), and in particular, the complete convex geometric graph on $2m$ vertices, denoted by $CGG(2m)$.

Definition 1.1 *A simple perfect matching (SPM) in $CGG(2m)$ is a set of m pairwise disjoint edges (i.e., edges that do not intersect, not even in an interior point).*

In [3], Keller and Perles give a complete characterization of the smallest sets of edges in $CGG(2m)$ that meet all the SPMs, called *blockers*. It turns out that all the blockers are simple trees of size m admitting a special structure called *caterpillar graphs* [1, 2], and that their number is $m \cdot 2^{m-1}$.

Following the result of [3], one may consider a sequence $\{A_n\}_{n=0}^{\infty}$, defined inductively as follows. A_0 is the family of all SPMs in $CGG(2m)$. Given A_k , define A_{k+1} as the family of all smallest sets of edges in $CGG(2m)$ that meet all of the elements of A_k . In particular, A_1 is the family of all blockers, characterized in [3].

A standard argument shows that $A_3 = A_1$, and thus $A_k = A_{k-2}$ for all $k \geq 3$. Thus, the only unknown element of the sequence is A_2 , i.e., the family of all smallest sets of edges of $CGG(2m)$ that meet all blockers, called in the sequel *co-blockers*. It is easy to show (see Section 3) that the size of any co-blocker is at least m , and on the other hand, any SPM meets every blocker by the definition of a blocker, and thus is a co-blocker. Therefore, the size of the co-blockers is m (like the size of the SPMs and of the blockers).

In this paper we give a complete characterization of the family of co-blockers:

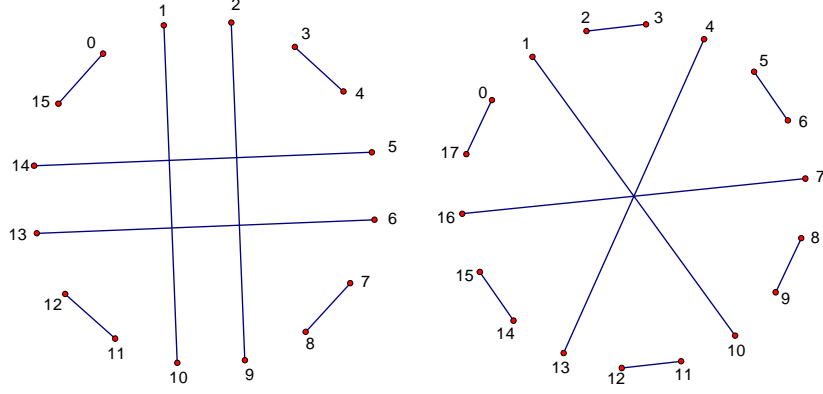


Figure 1: Two small co-blockers that are not SPMs.

Theorem 1.2 *For any $m \in \mathbb{N}$, the set of co-blockers in $CGG(2m)$ is the set of all perfect matchings in $CGG(2m)$, such that:*

- *All the edges of the matching have odd order (see Section 2 for a formal definition of the order of an edge in $CGG(2m)$).*
- *Two edges $[a, b]$ and $[a', c]$ of the matching whose end-points a, a' form a boundary edge of $CGG(2m)$ never cross.*

Two examples of small co-blockers that are not SPMs are given in Figure 1.

The theorem allows us to give lower and upper bounds on the number of co-blockers:

Proposition 1.3 *Denote the set of co-blockers in $CGG(2m)$ by $A_2(m)$. Then for all $m \in \mathbb{N}$,*

$$\lfloor m/3 \rfloor! \leq |A_2(m)| \leq m!.$$

It is known that both the number of SPMs and the number of blockers grow only exponentially with m : it is easy to show that the number of SPMs is the Catalan number $C_m = \frac{1}{m+1} \binom{2m}{m}$, and it is shown in [3] that the number of blockers is $m \cdot 2^{m-1}$. Thus, Proposition 1.3 shows that the number of co-blockers is significantly larger than the numbers of SPMs and blockers.

The rest of this paper is organized as follows: In Section 2 we introduce some basic definitions and recall the properties of SPMs and blockers that are used in our proof. In Section 3 we present the proof of Theorem 1.2. Finally, in Section 4 we prove Proposition 1.3.

2 Preliminaries

In this section we introduce several basic definitions, and recall some properties of SPMs and of blockers presented in [3], which are used in the proof of Theorem 1.2.

2.1 Definitions and Notations

Throughout this paper, we use the following definitions and notations.

Notation 2.1 *The set of vertices of $CGG(2m)$ is denoted by V , and is realized in the plane as the set of vertices of a convex $2m$ -gon P . The vertices are labelled cyclically from 0 to $2m - 1$.*

Definition 2.2 The **order** of an edge $[i, j]$ is $\min(|j - i|, 2m - |j - i|)$. The boundary edges of P are, of course, of order 1. We call the non-boundary edges, i.e., the edges that are diagonals of P , **interior edges**.

Definition 2.3 The **direction** of an edge in $CGG(2m)$ is the sum (modulo $2m$) of the labels of its endpoints. That is, if $e = [i, j]$, then its direction is:

$$\text{Dir}(e) = i + j \pmod{2m} = \begin{cases} i + j, & i + j < 2m \\ i + j - 2m, & i + j \geq 2m. \end{cases}$$

Two edges e, e' are **parallel** if $\text{Dir}(e) = \text{Dir}(e')$.¹

Definition 2.4 Two edges e, e' of $CGG(2m)$ are called **neighbors** if (at least) one endpoint of e is adjacent to (at least) one endpoint of e' on the boundary of P .

Definition 2.5 A perfect matching M of $CGG(2m)$ is called **semi-simple** if:

- All the edges of M are of odd order, and
- M does not contain a pair of crossing neighbors.

2.2 Caterpillar Trees and the Structure of Blockers

Definition 2.6 A tree T is a **caterpillar** (or a fishbone) if the derived graph T' (i.e., the graph obtained from T by removing all leaves and their incident edges) is a path (or is empty). A geometric caterpillar is **simple** if it does not contain a pair of crossing edges. A longest path in a caterpillar T is called a **spine** of T . Given a spine of T , the edges of T that have one endpoint interior to the spine and the other endpoint exterior to the spine are called **legs** of T .

In [3], the blockers in $CGG(2m)$ are fully characterized by the following theorem:

Theorem 2.7 Any blocker of $CGG(2m)$ is a simple caterpillar graph whose spine lies on the boundary of P and is of length $t \geq 2$. If the spine “starts” with the vertex 0 and the edge $[0, 1]$, then the edges of the blocker are:

$$\{[i - 1, i] : 1 \leq i \leq t\} \cup \{[t + j - 1 - \epsilon_{t+j}, t + j + \epsilon_{t+j}] : 1 \leq j \leq m - t\}, \quad (1)$$

where the ϵ_i are natural numbers satisfying $1 \leq \epsilon_{t+1} < \epsilon_{t+2} < \dots < \epsilon_m \leq m - 2$.

Conversely, any set of m edges of the described form is a blocker in $CGG(2m)$.

The contents of Formula (1) can be described as follows:

1. For each pair e, e' of opposite boundary edges of P , the blocker contains exactly one edge parallel to e and e' .
2. Each leg of the caterpillar connects a vertex a interior to the spine to a vertex b exterior to the spine.

¹Note that if P is regular, an equivalent definition is that e, e' are parallel as straight line segments in the plane.

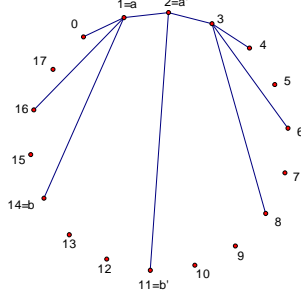


Figure 2: A blocker in $CGG(18)$.

3. If $[a, b]$ and $[a', b']$ are two distinct legs of the caterpillar (where a, a' are on the spine and b, b' are not), then the distance between b and b' along the complement of the spine is larger than the distance between a and a' (within the spine). In other words, if the spine starts with the vertex 0 and $a < a'$, then

$$b - b' > a' - a \quad (2)$$

(see Figure 2).

An example of a blocker in $CGG(18)$ is depicted in Figure 2.

In our proof we also use the following simple claim:

Claim 2.8 *In $CGG(2m)$, the set of all edges of odd order emanating from a single vertex is a blocker. This blocker is called “a star blocker”.*

The star blockers correspond to the case $t = 2$ in Theorem 2.7. The other extreme value $t = m$ yields blockers that are just one half of the boundary circuit of P .

3 Proof of Theorem 1.2

In this section we present the proof of our main theorem. We start by observing a simple necessary condition for co-blockers. As we shall see later, this condition is not so far from being sufficient.

Lemma 3.1 *Let C be a co-blocker in $CGG(2m)$. Then C is a perfect matching, and all the edges of C are of odd order.*

Proof: First, note that if there exists a vertex x that is not contained in any edge of C , then C does not meet the star blocker emanating from x , contradicting the assumption that C is a co-blocker. Thus, any vertex $x \in V$ is contained in an edge of C , and since C has only m edges (as noted in the introduction), this implies that C is a perfect matching.

Second, suppose on the contrary that C contains an edge $e = [x, y]$ of even order. Since e is the only edge of C that emanates from x , we again find that C does not meet the star blocker emanating from x , contradicting the assumption. Thus, all edges of C are of odd order. \square

We proceed by observing a property of semi-simple perfect matchings which will be crucial in our analysis:

Lemma 3.2 *Let M be a semi-simple perfect matching in $CGG(2m)$. Then the following holds:*

- *If e is an interior edge in M , then M contains a boundary edge in each of the two open half-planes determined by the straight line $\text{aff}(e)$.*
- *If e_1, e_2 are two crossing edges of M (i.e., edges which intersect in an interior point), then M contains a boundary edge in each of the four open quadrants determined by $\text{aff}(e_1)$ and $\text{aff}(e_2)$.*

Proof: We begin with the first claim. Let $e = [a, b]$ and let H be one of the half-planes determined by $\text{aff}(e)$. H meets the boundary of P in a polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$, where $x_0 = a$ and $x_k = b$. Consider the set of all edges of M both of whose endpoints are in $\{x_0, x_1, \dots, x_k\}$, like e . Among those edges, choose an edge $e' = [x_i, x_j]$ ($i < j$) that minimizes the difference $j - i$. We claim that e' is a boundary edge.

Indeed, if e' is not a boundary edge, then x_{i+1} is an internal vertex of the polygonal arc $\langle x_i, x_{i+1}, \dots, x_j \rangle$. Let e'' be the edge of M that contains x_{i+1} . By the minimality of e' , the other endpoint of e'' cannot be in $\{x_i, x_{i+1}, \dots, x_j\}$, and thus, e' and e'' are crossing neighbors in M , contradicting the assumption that M is semi-simple. Hence, e' is indeed a boundary edge, as asserted.

Now we proceed to the second claim. Let $e_1 = [a, b]$, $e_2 = [c, d]$, and $e_1 \cap e_2 = \{z\}$. Let Q be the quadrant determined by the rays \overrightarrow{za} and \overrightarrow{zc} . Q meets the boundary of P in a polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$, where $x_0 = a$ and $x_k = c$. We proceed by induction on k . The case $k = 1$ is impossible, since otherwise e_1 and e_2 are crossing neighbors, which contradicts the assumption that M is semi-simple.

Thus, we may assume that $k \geq 2$, and, in particular, that x_1 is an internal vertex of the polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$. Let $e' = [x_1, y]$ be the edge of M that contains x_1 . We consider four cases:

- If y is in $\{x_0, x_1, \dots, x_k\}$, then by the first claim, M contains a boundary edge e'' both of whose endpoints are in $\{x_1, x_2, \dots, x_{k-1}\}$, hence $e'' \subset \text{int}(Q)$.
- If y is on the boundary of P strictly between x_k and b , then the edge $[x_1, y]$ crosses the edge $[c, d]$ at some point $z' \in \text{int}(P)$ (see Figure 3). The quadrant Q' determined by the rays $\overrightarrow{z'x}$ and $\overrightarrow{z'c}$ meets the boundary of P in a shorter polygonal arc $\langle x_1, x_2, \dots, x_k \rangle$. Thus, by the induction hypothesis, M contains a boundary edge in $\text{int}(Q')$, and that edge is (of course) contained in $\text{int}(Q)$.
- If $y = b$, then M contains two edges emanating from the same vertex, contradicting the assumption that M is a perfect matching.
- If y is not one of the above, then the edges $[a, b]$ and $[x_1, y]$ are crossing neighbors, contradicting the assumption that M is semi-simple.

This completes the proof. \square

Now we are ready to state our main theorem.

Theorem 3.3 *Let M be a set of m edges in $CGG(2m)$. Then M is a co-blocker if and only if M is a semi-simple perfect matching.*

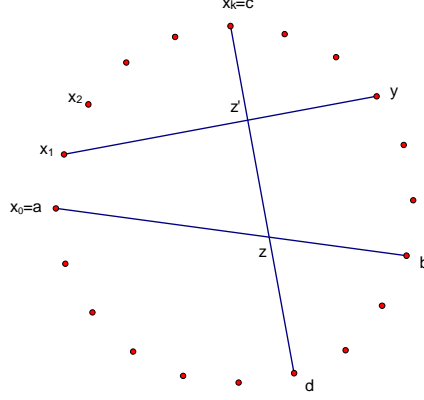


Figure 3: Illustration of the proof of Lemma 3.2.

Proof: Necessity: Assume that M is a co-blocker. By Lemma 3.1, M is a perfect matching and all its edges are of odd order. Suppose on the contrary that M is not semi-simple, and thus w.l.o.g. contains the crossing neighbors $[0, 2l - 1]$ and $[2k, 2m - 1]$, where $0 < 2k < 2l - 1 < 2m - 1$. Then the blocker B whose spine is $\langle 2m - 2, 2m - 1, 0, 1 \rangle$ and whose legs are $[0, 2j - 1]$ for all $2 \leq j < l$ and $[2j, 2m - 1]$ for all $l \leq j < m - 1$ does not meet M , contradicting the assumption that M is a co-blocker. The blocker B is depicted in Figure 4.

Sufficiency: Assume M is a semi-simple perfect matching, and suppose on the contrary that M misses some blocker B . Assume, without loss of generality, that the spine of B is $\langle 0, 1, 2, \dots, t \rangle$, where $2 \leq t \leq m$. (Note that by Theorem 2.7, the blocker is a caterpillar whose spine lies on the boundary of P .) For $i = 1, 2, \dots, t - 1$, denote by e_i the (unique) edge of M that emanates from i , and denote its other endpoint by y_i . We claim that the edges e_1, e_2, \dots, e_{t-1} satisfy the following:

- For any $1 \leq i \leq t - 1$, we have $y_i \in \{t + 1, t + 2, \dots, 2m - 1\}$. In particular, the $t - 1$ edges e_1, e_2, \dots, e_{t-1} are distinct.
- For any pair $i, j \in \{1, 2, \dots, t - 1\}$, the edges e_i and e_j do not cross.

The first claim follows from the first claim of Lemma 3.2. Indeed, if $y_i \in \{0, 1, \dots, t\}$, then by the lemma, M contains a boundary edge in the polygonal path between i and y_i , and by assumption, this edge is in the spine of B , which contradicts the assumption that M misses B . The second claim follows similarly from the second claim of Lemma 3.2.

The second claim implies that if $1 \leq i < i + 1 \leq t - 1$, then $y_{i+1} < y_i$, and therefore, $i + y_i \geq (i + 1) + y_{i+1}$. Thus, the function $g : i \mapsto i + y_i$ is monotone non-increasing in i for $1 \leq i \leq t - 1$. We can also bound the range of this function, namely,

$$2m - 1 = 1 + (2m - 2) \geq g(1) \geq g(i) \geq g(t - 1) \geq (t - 1) + (t + 2) = 2t + 1,$$

which implies that for all $1 \leq i \leq t - 1$,

$$\text{Dir}(e_i) = i + y_i \pmod{2m} = i + y_i = g(i).$$

Since, by Theorem 2.7, the blocker B contains a unique edge parallel to every edge of odd order in $CGG(2m)$, there is a unique edge f_i of B parallel to e_i . This edge cannot lie on the spine

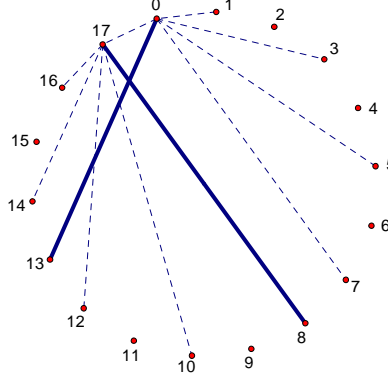


Figure 4: A blocker that misses a perfect matching with crossing neighbors.

of B , since for the edges on the spine of B , the direction takes the values $1, 3, \dots, 2t - 1$, and thus they are not parallel to the edges e_i . Hence, f_i is a leg of B , which can be represented as $f_i = [r_i, q_i]$, with $1 \leq r_i \leq t - 1$, and $t + 1 \leq q_i \leq 2m - 1$.

Note that we have $Dir(f_i) = r_i + q_i$. (The other option, $Dir(f_i) = r_i + q_i - 2m$, would yield $Dir(f_i) \leq t - 2$, whereas $Dir(f_i) = Dir(e_i) \geq 2t + 1$.) Thus, by the monotonicity of $Dir(e_i)$, we have:

$$r_{i+1} + q_{i+1} = Dir(f_{i+1}) = Dir(e_{i+1}) \leq Dir(e_i) = Dir(f_i) = r_i + q_i, \quad (3)$$

for $i = 1, 2, \dots, t - 2$.

Now we are ready to reach the contradiction. If $r_i = i$ for some i , then $f_i = e_i$, contrary to the assumption that $M \cap B = \emptyset$. Thus, $r_1 > 1$ and $r_{t-1} < t - 1$. Hence, there are two consecutive indices $1 \leq i < i + 1 \leq t - 1$ with $r_i > i$ and $r_{i+1} < i + 1$. It follows that $r_{i+1} < r_i$, and therefore, by Theorem 2.7, we must have $q_{i+1} - q_i > r_i - r_{i+1}$ (see Equation (2)), which contradicts Equation (3). This completes the proof. \square

4 The Number of Co-Blockers

The characterization of the co-blockers given in Theorem 1.2 allows us to find upper and lower bounds on the number of co-blockers in $CGG(2m)$, as a function of m .

Proposition 4.1 *The number of co-blockers satisfies*

$$\lfloor m/3 \rfloor! \leq |C(m)| \leq m!. \quad (4)$$

Proof: The right inequality in (4) follows immediately from Lemma 3.1, since by the lemma, all the co-blockers are perfect matchings whose edges are of odd order. These matchings can be viewed as bijections from the set of vertices of odd index to the set of vertices of even index, and their number is $m!$.

In order to prove the left inequality in (4), we consider perfect matchings of a special type. For the sake of simplicity we assume first that $m = 3k$, and denote the vertices of $CGG(2m)$ by $0, 1, 2, \dots, 2m - 1$. We consider only perfect matchings that contain all the boundary edges

$$[1, 2], [4, 5], \dots, [6k - 5, 6k - 4], [6k - 2, 6k - 1],$$

i.e., all the boundary edges whose vertices are congruent to 1 and 2 modulo 3. We claim that any perfect matching of this class whose edges are all of odd order is a co-blocker. Indeed, by Theorem 1.2, such a perfect matching is not a co-blocker only if it contains two edges whose endpoints are consecutive vertices on the boundary which intersect in an interior point. However, amongst any two consecutive vertices on the boundary there is a vertex whose index modulo 3 equals to 1 or 2, and thus the edge of the matching containing that vertex is a boundary edge and cannot cross any other edge.

The number of perfect matchings of this class is $k!$, since any vertex whose index equals 0 modulo 6 can be connected by an edge to any vertex whose index equals 3 modulo 6, and all the other vertices are already contained in boundary edges.

Thus, $|C(m)| \geq k! = (m/3)!$. Finally, if $m = 3k + 1$ or $m = 3k + 2$, then one may consider perfect matchings of the class described above, but containing also the boundary edge $[6k, 6k+1]$ (for both $m = 3k + 1, m = 3k + 2$), and in addition $[6k + 2, 6k + 3]$ (for $m = 3k + 2$). The argument given above in the case $m = 3k$ applies also here, and the number of such perfect matchings is $k!$. Thus, $|C(m)| \geq \lfloor m/3 \rfloor!$, as asserted. \square

References

- [1] F. Harary and A.J. Schwenk, Trees with Hamiltonian Square, *Mathematika* **18** (1971), pp. 138-140.
- [2] F. Harary and A.J. Schwenk, The Number of Caterpillars, *Disc. Math.* **6** (1973), pp. 359–365.
- [3] C. Keller and M. Perles, On the Smallest Sets Blocking Simple Perfect Matchings in a Convex Geometric Graph, *Israel J. of Math.*, in press. Available on-line at <http://arxiv.org/abs/0911.3350>.